

An algebraist's guide to Thom spectra

Variations on an oriental folksong

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The notes were roughly proofread.
There shouldn't be substantial mistakes.
I'm not planning on adding more material.
Feedback on the math and the storytelling is welcome.

The present notes largely grew out of me reading [ABGHR14]. The mathematical contents are hardly if ever original, as most of the effort is in the interpretation into the present language.

In other words, this is my retelling of a classical story from a 2020s' algebraist's perspective.

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Prelude

Let M be a closed smooth m -manifold and ξ a vector bundle on M . One might want to find invariants of ξ , or to study the zero locus of a generic section. Quite often one finds the question encoded through the Thom construction of ξ . Classically this is presented in the literature in a multitude of fashions:

1. $\text{Th}\xi$ is the one-point compactification of the total space $E\xi$.
2. With a choice of bundle metric on ξ , one can identify $\text{Th}\xi$ with the quotient $D\xi/S\xi$ of the disk bundle by the sphere bundle. Equivalently, it is the mapping cone of the projection from the sphere bundle $\text{cofib}(S\xi \rightarrow M)$.
3. Without making the contractible choice of metric, one may replace the sphere bundle in the above construction with the bundle of non-zero vectors. This gives $E\xi/(E\xi \setminus 0_M)$.

We algebraists immediately dismiss the first construction as nonsense, and indeed it should be viewed as a special case of 2. when M is compact. The 3rd description best reflects the geometric intuition in that the (co)homology of $\text{Th}\xi$ is the local (co)homology at the zero section.

This note reinterprets Thom’s construction in the “*Fundamental Six Functors of Algebraic Topology*”, and deduce some classical results of algebraic topology.

$$\begin{array}{ccc} \text{An} & \longrightarrow & \text{Pr}_{\text{st}}^{\text{L}} \\ (X \xrightarrow{f} Y) & \longmapsto & (\text{Sp}^X \xrightleftharpoons[f^*]{f_!} \text{Sp}^Y). \end{array}$$

In a sense, this is an instance of the move from “classical” towards “abstract” topology, and a demonstration of the separation of “algebra” from “topology”.

Caveat! I’m not establishing the category theory rigorously in these notes. My actual motive is to convey to the reader the appeal of an algebraic perspective to topology. Six functor formalisms originated in algebraic geometry, which provides most of the relevant examples. I hope to convince the reader that this language can be useful to them even if they aren’t motivated by algebraic geometry.

For the sections relevant to manifolds, I also chose to hide the analysis and differential geometry away from the exposition.

- I’ll restrict our attention to (perhaps connected or with boundary) compact manifolds. The homotopy datum we need is the *pair* $(M, \partial M)$, i.e. the inclusion $\partial M \hookrightarrow M$, which for closed manifolds reduces to the anima $M : \text{An}$ or $M_+ : \text{An}_*$.

- The connection back to geometry is achieved through *Atiyah duality* 2.8 which relates differential geometric datum (namely the tangent bundle) with the stable homotopy type of M . Up to knowledge about embeddings of manifolds, Atiyah duality can be established in a mostly formal manner, see [Rez13].

There's an nLab article on the uniqueness of embeddings. Go read it.

- I won't discuss the role Thom spectra play in bordism theory. Some special cases are known under the name of Madsen–Tillman spectra. The curious reader would have already found themselves looking up these notions.

Last but not least, let's bid a farewell to the construction of Thom spaces by one-point compactification. Take a vector bundle ξ over a finite CW base X so that $\text{Th}(\xi) \simeq E\xi \text{ "U" } \infty$. If we can identify $E\xi$ as the open complement $Y \setminus Z$ for a CW pair (Y, Z) , we obtain an identification

$$\text{Th}(\xi) = D\xi/S\xi \simeq E\xi \text{ "U" } \infty \simeq (Y \setminus Z) \text{ "U" } \infty \simeq Y/Z.$$

An actual application of this identification is the following:

Exercise 1 Let σ_n be the tautological bundle over the real projective space $\mathbb{R}P^n$, then one can identify the Thom space of $k\sigma_n$

$$\text{Th}(k\sigma_n) \simeq \mathbb{R}P_k^{n+k} := \mathbb{R}P^{n+k}/\mathbb{R}P^{k-1}$$

as a *stunted projective space*¹ given above.

Conventions

- The symbol $:=$ indicate a definition, whereas $=$ indicates an equality. The former works at the level of English and not at the level of mathematical statements the same way the latter does. The same goes for \equiv .

It's hard to explain the difference between $=$ and \simeq , or whether there is a difference at all.

- The phrase “ n -category” refers to (∞, n) -categories. A category is a 1-category.
- We generically avoid set-theoretic notation and write $x : C$ instead of $x \in C$ for the declaration of x to be of type C , e.g. x be an object or morphism of a category C . (To be pedantic, of type C^\approx , the groupoid core of C resp. $(C^{[1]})^\approx$).

The notation $- \in A$ would rarely occur as a predicate for a vague notion of subtyping. For instance, if x is an object or morphism in a category C , with a subcategory C_0 , then $c \in C_0$ means “ c belongs to the subcategory C_0 ”.

- We fix $k \in \{\mathbb{R}, \mathbb{C}\}$ ² and denote the general linear group by $GL(n) \in \{O(n), U(n)\}$.
- The final object of a category \mathcal{C} is denoted $1_{\mathcal{C}}$ or 1 , and the unique map from $x : \mathcal{C}$ to 1 is also denoted $x : x \rightarrow 1$.

¹I'm quite uncomfortable that the subscript k indicates quotienting out by $\mathbb{R}P^{k-1}$. In a later exercise I will attempt to justify the notation $\Sigma_+^k \mathbb{R}P^n$.

²This usage of \in here is meta-theoretical and hence doesn't contradict my convention.

- The word “ring” refers to E_1 -algebras in Sp , the category of spectra, or more generally E_n -algebras in any stably (symmetric) monoidal category.

Discrete rings (E_1 -algebras in the category Ab of abelian groups) are view as rings through their Eilenberg–Mac Lane spectra, where we omit H and write \mathbb{Z} etc.

- The tensor product \otimes can be decorated in different ways. Tensor products of categories are always the Lurie tensor product in Pr^L . In a category, the notations \otimes_X and \otimes^X denote the fiberwise and convolution monoidal structure on Sp^X , see Proposition 1.15, part 3.

The tensor product of (module) spectra of over A is written \otimes_A . In the parametrised setting $\otimes_{A,X}$ resp. \otimes_A^X . For $A = \mathbb{S}$ I omit \mathbb{S} when $X \neq 1$ to keep the notations easier to read (i.e. \otimes_X resp. \otimes^X).

Suggestions for better notations are welcome. What I can think of:

- Decorate fiberwise as $- {}_A \otimes_X - / - {}_X \otimes_A - / - \otimes_{A_X} -$.
- \otimes_A^X for fiberwise and $\circlearrowleft_A^X / \circlearrowright_A^X$ for convolution.
- At a few places I use the undecorated \otimes for the tensoring of presentable categories over *anima* in Sp , so that for $X : \mathrm{An}$ and $A : \mathrm{Sp}$, the notation $A \otimes X$ denotes $A \otimes_{\mathbb{S}} \Sigma_+^\infty X$. This serves to avoid writing Σ^∞ too often.

Another (better) notation is to write $A[X]$ for $A \otimes X$ as above. Similarly, we can write the tensoring by a pointed space (X, x) as $A[X, x]$. This is mainly used in Appendix A.

Acknowledgement

I'd like to thank Elia Auer and Bastiaan Cnossen for comments. **The notes are not finalised.** Your feedback is welcome!

1. Thom spaces: the algebra of vector bundles

This section is hugely rephrasing [ABGHR14].

The geometric idea behind the Thom construction can be tentatively summarised as follows:

Definition 1.1 (Thom space: geometric intuition) *Given a \mathbb{k} -vector bundle E over the anima $X : \mathbf{An}$, we form the **fiberwise one-point compactification** of E :*

$$S_X^E := \sum_{x:X} S^{E_x}, \quad \text{where } S^{E_x} := E_x \cup \infty_x.$$

We regard S_X^E as a bundle of spheres pointed at

$$\infty_X : X \rightarrow S_X^E, x \mapsto \infty_x.$$

This construction is functorial with respect to inclusion of subbundles. In particular, there is a canonical map of pointed bundles coming from the inclusion of the zero bundle:

$$0_X : S_X^0 \simeq X \coprod X \xrightarrow{\langle 0_X, \infty_X \rangle} S_X^E.$$

The **Thom space** of E is the (homotopy) cofiber:

$$\mathrm{Th}_X(E) := S_X^E / \infty_X$$

equipped with the canonical zero section

$$(0_X)_+ : X_+ \simeq \mathrm{Th}_X(0) \rightarrow \mathrm{Th}_X(E).$$

1.1. The algebra of bundles: (Un)straightening

The following fundamental result of category theory allows us to view *bundles of spaces*, or *parametrised spaces* as functors. It will prove useful later where the parametrised objects aren't spaces anymore.

Definition-Proposition 1.2 ((Un)straightening of spaces) *The category of **parametrised spaces**, or **bundles** over $X : \mathbf{An}$ is the functor category \mathbf{An}^X . The **(un)straightening equivalence** states that*

$$\tau : \mathbf{An}/X \longrightarrow \mathrm{Fun}(X, \mathbf{An})$$

$$\left(\begin{array}{c} E \\ \downarrow \\ X \end{array} \right) \longmapsto \left(\begin{array}{ccc} x & \mapsto & E_x \quad \text{the fiber over } x \\ (x \xrightarrow{\gamma} y) & \mapsto & (E_x \xrightarrow{\tau_\gamma} E_y) \quad \text{the fiber transport along } \gamma \end{array} \right)$$

is an equivalence of categories.

The category of **pointed bundles** over X is the category of pointed objects in \mathbf{An}/X . Equivalently, it is the over-under-category $\mathbf{An}_{X/X}$ and the functor category \mathbf{An}_*^X .

The algebra of vector bundles follows a similar vein.

Definition 1.3 (Topological K-theory) *The topological category of finite dimensional \mathbb{k} -vector spaces and linear injections presents a category which we call \mathbf{vb}_{inj} . Its groupoid core is the **classifying space** of \mathbb{k} -vector bundles:*

$$\mathbf{vb} := (\mathbf{vb}_{inj})^{\simeq} \simeq \coprod_{n \geq 0} \mathbf{BGL}(n).$$

*Direct sum of vector spaces equips (\mathbf{vb}_{inj}) and \mathbf{vb} with the structure of an \mathbb{E}_∞ -monoid(al category). The group completion of (\mathbf{vb}, \oplus) is the **connective topological K-theory space***

$$\mathbf{kgl} := \mathbf{vb}^{grp} \simeq \mathbb{Z} \times \mathbf{BGL}.$$

The space of vector bundles (resp. virtual vector bundles) on $X : \mathbf{An}$ is the mapping space \mathbf{vb}^X (resp. \mathbf{kgl}^X).

Remark 1.4. Tensor product of vector bundles equips \mathbf{vb}^X with a \mathbb{E}_∞ -semiring structure. Consequently \mathbf{kgl}^X is a connective ring spectrum. This structure will be irrelevant to our story.

Note that we're in the topological setting, so \mathbf{BGL} is \mathbf{BO} or \mathbf{BU} , and no $+$ -construction is necessary.

Definition 1.5 (The associated sphere bundle) *The **unit sphere** functor $\mathbf{Sph} : \mathbf{vb}_{inj} \rightarrow \mathbf{An}$ is the functor presented by the topological functor sending a vector space V to $V \setminus 0$.*

*The **associated sphere** functor $\mathbf{S}^- : \mathbf{vb}_{inj} \rightarrow \mathbf{An}_*$ is the functor presented by the topological functor sending V to $V \cup \infty$ based at ∞ .*

For anima X , we call the functors

$$\begin{aligned} \mathbf{Sph}_X & : \mathbf{vb}_{inj}^X \longrightarrow \mathbf{An}^X \simeq \mathbf{An}/X \\ \mathbf{S}_X^- & : \mathbf{vb}_{inj}^X \longrightarrow \mathbf{An}_*^X \simeq \mathbf{An}_{X//X} \end{aligned}$$

*the **unit sphere bundle** resp. the **associated sphere bundle**.*

The associated sphere is equivalent to the unpointed suspension of the unit sphere (as the left pushout below). As such it is equipped with two canonical cone points 0_X and ∞_X .

This leads to our algebraic framing of the geometric intuition at the beginning of this section.

Definition 1.6 (Thom space: algebraic intuition) *For $E : \mathbf{vb}_{inj}^X$, consider the morphism $0 \rightarrow E : \mathbf{vb}_{inj}^X$ and the induced morphism in \mathbf{An}_*^X between their associated spheres $0_X : \mathbf{S}_X^0 \xrightarrow{0_X} \mathbf{S}_X^E$.*

*The **Thom space** of E is the integration (i.e. left Kan extension) of this map along the final map $X \rightarrow 1$.*

$$X_+ \simeq X_!(\mathbf{S}_X^0) \xrightarrow{X_!(0_X)} X_!(\mathbf{S}_X^E) =: \mathbf{Th}_X(E). \quad (\text{in } \mathbf{An}_*)$$

We sometimes abuse language to speak of the codomain $\mathbf{Th}_X(E)$ as the Thom space.

Equivalently, the Thom space is the lower row in the following pushout in \mathbf{An} :

$$\begin{array}{ccccc} \mathbf{Sph}_X(E) & \xrightarrow{\pi} & X & \longrightarrow & 1 \\ \downarrow \pi & & \downarrow \infty_X & \lrcorner & \downarrow \\ X & \xrightarrow{0_X} & \mathbf{S}_X^E & \longrightarrow & \mathbf{Th}_X(E). \end{array}$$

where π is the projection of the unit sphere bundle. Note that the left square is a suspension in the topos \mathbf{An}/X .

Exercise 2 Convince yourself that this definition coincides with the tentative definition 1.1.

The associated sphere functor is compatible with the monoidal structures on vb_{inj} and An_* :

Proposition 1.7 *The functor $S^- : \text{vb}_{\text{inj}} \rightarrow \text{An}_*$ is symmetric monoidal: It sends $V \oplus W$ to $S^V \wedge S^W$. \square*

This is easily promoted into a parametrised version, but we'll have to first prepare the stage a bit.

Definition 1.8 (Pullback of bundles) *For $X \xrightarrow{f} Y : \text{An}$, the pullback along f of a vector bundle $\xi : \text{vb}_{\text{inj}}^Y$ or a pointed bundle $E : \text{An}_*^Y$ is*

$$\begin{aligned} f^* \xi &: X \xrightarrow{f} Y \xrightarrow{\xi} \text{vb}_{\text{inj}} \quad \text{resp.} \\ f^* E &: X \xrightarrow{f} Y \xrightarrow{E} \text{An}_*. \end{aligned}$$

Pre- and postcomposition commute, whence we deduce the following.

Proposition 1.9 *Pullback and associated spheres commute, i.e. $f^* S_Y^\xi \simeq S_X^{f^* \xi}$. \square*

We also need a notion of sums of vector bundles. In this setting it's helpful to think of spaces over a base Z as spaces parametrised by Z .

Definition 1.10 (Direct sums and smash products) *Consider a pullback $W := X \times_Z Y$ as follows.*

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ \downarrow g' & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

For $\zeta : \text{vb}_{\text{inj}}^X$, $\xi : \text{vb}_{\text{inj}}^Y$, their **external direct sum relative Z** is defined as

$$\zeta \boxplus_Z \xi : W = X \times_Z Y \xrightarrow{[\zeta] \times_Z [\xi]} \text{vb} \times \text{vb} \xrightarrow{\oplus} \text{vb}.$$

Similarly for $E : \text{An}_*^X$, $L : \text{An}_*^Y$ we define the **external smash product relative Z** of to be

$$E \boxtimes_Z L : W = X \times_Z Y \xrightarrow{E \times_Z L} \text{An}_* \times \text{An}_* \xrightarrow{\wedge} \text{An}_*.$$

In the case $Z = 1$ we refrain from saying “relative 1”.

In case $X = Y$ we define their **internal direct sum** resp. **internal smash product** as the pullback of the external ones along the diagonal $X \rightarrow X \times X$:

$$\begin{aligned} \zeta \oplus_X \xi &: X \xrightarrow{(\zeta, \xi)} \text{vb} \times \text{vb} \xrightarrow{\oplus} \text{vb}. \\ E \wedge_X L &: X \xrightarrow{(E, L)} \text{An}_* \times \text{An}_* \xrightarrow{\wedge} \text{An}_*. \end{aligned}$$

Remark 1.11. In the situation above, the external direct sum can be described using the internal direct sum on $W = X \times_Z Y$ as

$$\zeta \boxplus_Z \xi \simeq g'^*(\zeta) \oplus_W f'^*(\xi).$$

The analogous statement holds for pointed bundles.

These are part of a symmetric monoidal structure on the relevant categories, more discussion of which will take place after section 1.2. For now we content ourselves with the following consequence of the monoidality of $S^- : \mathbf{vb}_{\text{inj}} \rightarrow \mathbf{An}_*$.

Lemma 1.12 *In the situation above, the external direct sum and smash product are compatible in that*

$$S_W^{\zeta \boxplus_Z \xi} \simeq S_X^\zeta \boxtimes_Z S_Y^\xi : \mathbf{An}_*^W.$$

Consequently for the internal direct sum and fiberwise smash product:

$$S_X^{\zeta \oplus_X \xi} \simeq S_X^\zeta \wedge_X S_X^\xi : \mathbf{An}_*^X. \quad \square$$

Remark 1.13. The correct setting to work with these is the slice category construction, which functorially takes a monoid \mathfrak{a} in a category \mathcal{C} to a monoidal category $\mathcal{C}/_{\mathfrak{a}}$. The construction is in [Lur17, Sec. 2.2.2]. See also Maxime Ramzi’s paper [Ram26] on the compatibility with unstraightening.

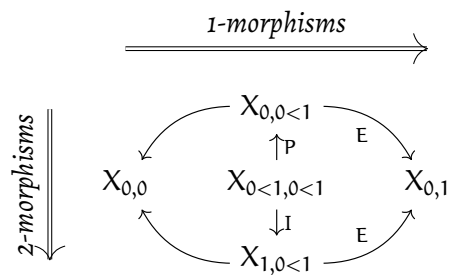
The Thom space and related Thom spectra are in a sense the “integration” of these parametrised spheres over the base space. We take this as an excuse to introduce the language of six functors, as they provide a much convenient framework to talk about these operations.

1.2. The six functors

In order to clarify the various properties of Thom construction, we introduce the omnipresent **six functor formalisms** in geometry. One often treats them within the framework of *symmetric monoidal stable categories*, but much of what we need can be found at the level of the pointed category \mathbf{An}_* .

Almost all of this section is borrowed from Cnossen–Lenz–Linskens [CLL25].

Definition 1.14 (Six functor formalisms, cf. [CLL25]) *For a suitable “geometric setting”³ $\mathcal{C}_{E,I,P}$, one can define a symmetric monoidal span 2-category $2\text{Span}(\mathcal{C}_{E,I,P})$, whose objects are the same as \mathcal{C} , and the morphism categories are given by (I_+, P^\dagger) -spans between E_+ -spans*



and composition by pullback, equipped with a symmetric monoidal structure given by products in \mathcal{C} .

³A category \mathcal{C} with wide subcategories $\mathcal{C} \supseteq E \supseteq I, P$ subject to various axioms. See [CLL25, Thm. A and B]. For now, just think of $\mathbf{An}_{\text{all,all,iso}}$.

A (presentable) **six functor formalism** on S

$$D : \begin{array}{ccc} C^{op} & \longrightarrow^{E,I,P} & \mathbf{Pr}^L \\ (X \xrightarrow{f} Y)^{op} & \longmapsto & (D(X) \xrightleftharpoons[f_!]{f^*} D(Y)), \end{array}$$

is a lax symmetric monoidal functor⁴

$$D : \begin{array}{ccc} 2\text{Span}(C_{E,I,P}) & \longrightarrow & \mathbf{Pr}^L \\ X & \longmapsto & D(X) \\ \left(\begin{array}{ccc} & Y & \\ f \swarrow & & \searrow g \\ X & & Z \end{array} \right) & \longmapsto & \left(\begin{array}{ccc} & D(Y) & \\ f^* \nearrow & & \searrow g_! \\ D(X) & & D(Z) \end{array} \right) \\ \dots & \longmapsto & \dots \end{array}$$

This (relatively) concise⁵ definition encodes an enormous amount information. We unpack some of them.

Proposition 1.15 (The magic scroll) A presentable six functor formalism $D : C^{op} \rightarrow^{E,I,P} \mathbf{Pr}^L$ contains (at least) the following information:

- (Adjoints) For any morphism $X \xrightarrow{f} Y$ in C , there is the pullback $D(X) \xleftarrow{f^*} D(Y)$. If $f \in E$ ⁶, there is the pushforward, or integration $D(X) \xrightarrow{f_!} D(Y)$. As morphism in \mathbf{Pr}^L , the functors admit right adjoints which we denote $f^* \dashv f_*$ resp. $f_! \dashv f^!$

If $f \in I$ then there is an (internal) adjunction $f_! \dashv f^*$ so that $f^! \simeq f^*$, and if $f \in P$ the opposite adjunction $f^* \dashv f_!$ resp. $f_* \simeq f_!$.

If $f \in I \cap P$ then one has an ambidextrous adjunction.

- (Base change and (I, P) -biadjointability): Consider a pullback diagram in C with $g \in E$:

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ \downarrow g' & \lrcorner & \downarrow g \\ Z & \xrightarrow{f} & W \end{array}$$

There is a base change isomorphism $\alpha : g'_! f'^* \simeq f^* g_!$, compatible with composition of pullback squares.

If $g \in I$ or $g \in P$, then α is given by the Beck–Chevalley transformation

$$BC : \begin{cases} g'_! f'^* \Rightarrow f^* g_! & g \in I \\ g'_! f'^* \Leftarrow f^* g_! & g \in P \end{cases} .$$

In particular, for any decomposition of $E \ni g = p \circ i$, $p \in P$, $i \in I$ ⁷, the base-change isomorphism is identified with the Beck–Chevalley transformations for p and i .

If $f \in P$ and $g \in I$, then the (double) Beck–Chevalley transformation $g'_! f'_! \Rightarrow f_! g'_!$ is also an equivalence.

- (Monoidal structures): The inclusions $(-)^{\dagger} : C^{op} \rightarrow 2\text{Span}(C_{E,I,P})$ and⁸ $(-)_+ : E \rightarrow 2\text{Span}(C_{E,I,P})$

⁴I might want it to preserve direct sums if C is disjunctive, like \mathbf{An} . Here the direct sum in $2\text{Span}(C_{E,I,P})$ is given by disjoint union. Funny thing that \mathbf{Pr}^L is semiadditive.

⁵??????

⁶I mean f , which has type “morphism of C ”, satisfies the predicate “belongs to the subcategory E ”.

⁷The existence of i is an axiom of a “geometric setting”.

⁸Assuming that E, I, P are suitably compatible with products.

are symmetric monoidal.

As any object X is canonically a Cartesian coalgebra through the diagonal, each $D(X)$ is canonically equipped with a symmetric monoidal structure called **(fiberwise) tensor product** \otimes_X . The monoidal unit is denoted 1_X . Pullback f^* is symmetric monoidal and f_* is lax symmetric monoidal. Integration $f_!$ is oplax monoidal iff $f \in \mathbf{I}$ and lax monoidal iff $f \in \mathbf{P}$.

If X is an \mathcal{O} -algebra in \mathbf{E} , then $D(X)$ is equipped with an \mathcal{O} -monoidal structure called **(Day) convolution** \otimes^X , whose monoidal unit we denote by δ^X in analogy to the “Dirac delta”. Integration $f_!$ is \mathcal{O} -monoidal and $f^!$ lax \mathcal{O} -monoidal along maps of \mathcal{O} -algebras, while f^* is lax monoidal iff $f \in \mathbf{I}$ and oplax monoidal iff $f \in \mathbf{P}$.

The internal mapping functor in either case will be denoted by

$$\underline{\text{map}}_X(-, -) \text{ resp. } \underline{\text{map}}^X(-, -) : D(X)^{\text{op}} \times D(X) \rightarrow D(X).$$

4. (Projection formula): In the Cartesian monoidal structure, any morphism $X \xrightarrow{f} Y$ is a morphism of commutative coalgebras, endowing X with a Y -comodule structure. Consequently pullback f^* makes $D(X)$ a commutative $D(Y)$ -algebra under tensor product.

If $f \in \mathbf{E}$ then $f_!$ is a morphism of $D(Y)$ -modules, i.e.:

$$f_!(-_1) \otimes_Y -_2 \simeq f_!(-_1 \otimes_X f^*(-_2)) \quad D(X) \otimes D(Y) \rightarrow D(Y)$$

whose right adjoints⁹ also coincide.

$$\begin{aligned} f^! \underline{\text{map}}_Y(-_1, -_2) &\simeq \underline{\text{map}}_X(f^*(-_1), f^!(-_2)) & D(Y)^{\text{op}} \times D(Y) &\rightarrow D(X) \\ \underline{\text{map}}_Y(f_!(-_1), -_2) &\simeq f_* \underline{\text{map}}_X(-_1, f^!(-_2)) & D(X)^{\text{op}} \times D(Y) &\rightarrow D(Y) \quad \square \end{aligned}$$

Remark 1.16. The projection formula is a special feature of the Cartesian monoidal structure which has no analogue for convolution. More precisely, the map $X \rightarrow Y$ of Y -comodules consists of Cartesian squares:

$$\begin{array}{ccc} X & \xrightarrow{(\text{id}_X, \delta_Y^{(n)} \circ f)} & X \times Y^n \\ f \downarrow \ulcorner & & \downarrow f \times \text{id}_{Y^n} \\ Y & \xrightarrow{(\text{id}_Y, \delta_Y^{(n)})} & Y \times Y^n \end{array}$$

In case $f \in \mathbf{E}$, integration $f_!$ being a morphism of $D(Y)$ -modules is precisely the Beck–Chevalley condition on the vertical left adjoints, of which the projection formula is the special case $n = 1$.

The universal property of the span 2-category addresses the unique extension of a given functoriality $(-)^*$ into a full-fledged six functor formalism.

Theorem 1.17 ([CLL25, Thm. A and B]) *In a suitable geometric setting $C_{\mathbf{E}; \mathbf{I}, \mathbf{P}}$, the contravariant inclusion $(-)^{\dagger} : C^{\text{op}} \rightarrow 2\text{Span}(C_{\mathbf{E}; \mathbf{I}, \mathbf{P}})$ is the initial (\mathbf{I}, \mathbf{P}) -biadjointable functor into a 2-category under C^{op} . In other words, any functor $C^{\text{op}} \rightarrow \mathbf{C}$ into a 2-category \mathbf{C} which admit the adjoints as **1** and are (\mathbf{I}, \mathbf{P}) -biadjointable as **2** extends uniquely to the span 2-category.*

A similar universal property holds with lax monoidality. □

⁹Remembering that $D(X) \otimes D(Y) \simeq \text{RFun}(D(X)^{\text{op}}, D(Y)) \simeq \text{RFun}(D(Y)^{\text{op}}, D(X))$ and how they relate to the internal mapping objects.

Remark 1.18 ([[CLL25](#), Prop. 5.16]). The universal property is stated in [[CLL25](#), Thm. B] with *lax* monoidality. As one can check, the locally full subcategories on the strong symmetric monoidal functors agree on both sides.

Take a rest! Six functors are ubiquitous in geometry and we won't be able to do justice to the subject. The reader shall read other people's writings such as [[Sch23](#)][[Gal25](#)][[Vol23](#)].

For the 2-categorical notions revolving around Beck–Chevalley, I can only point to [[HHLN23](#)] by Haugseng–Hebestreit–Linskens–Nuiten.

1.3. The algebra of Thom spaces

Taking presheaves is symmetric monoidal in *both* the restriction $\text{An}^{\text{op}} \xrightarrow{(\text{PShv})^*} \text{Pr}^L$ functoriality and the left Kan extension $\text{An} \xrightarrow{(\text{PShv})_!} \text{Pr}^L$ functoriality. They assemble into a six functor formalism.

Theorem 1.19 *There exist the six functor formalism of parametrised spaces, or the unstable “fundamental six functors of algebraic topology”:*

$$\begin{array}{ccc} \text{An}^{\text{op}} & \xrightarrow{\text{all}; \text{all}, \text{iso}} & \text{Pr}^L \\ (\text{X} \xrightarrow{f} \text{Y})^{\text{op}} & \longmapsto & (\text{PShv}(\text{X}) \underset{f_!}{\overset{f^*}{\rightleftarrows}} \text{PShv}(\text{Y})); \end{array}$$

where $f_! \dashv f^*$ is the adjunction of left Kan extension and restriction, which we also call integration and pullback based on intuition of bundles. (Two of the six functors are equivalent in this simplified setting, giving a five functor formalism)

The two symmetric monoidal embeddings of $(-)^{\dagger} : \text{An}^{\text{op}} \rightarrow 2\text{Span}(\text{An})$ and $(-)_+ : \text{An} \rightarrow 2\text{Span}(\text{An})$ provide two flavors of monoidal structures.

- **(The fiberwise monoidality)** The former equips every $\text{PShv}(\text{X})$ with the symmetric monoidal structure of fiberwise product \times_{X} with monoidal unit 1_{X} . Pullback f^* is symmetric monoidal, integration $f_!$ is oplax monoidal and satisfies the projection formula.
- **(The Day convolution monoidality)** The latter equips $\text{PShv}(\text{X})$ for \mathbb{E}_n -spaces X with the \mathbb{E}_n -monoidal structure of **Day convolution**¹⁰ \times^{X} . Along \mathbb{E}_n -maps f , integration $f_!$ is monoidal and f^* is lax monoidal. The monoidal unit δ^{X} is the integration of $1 : \text{An}$ along the unit morphism $1 \rightarrow \text{X}$.

$$\delta^{\text{X}}(x) = \text{map}_{\text{X}}(1_{\text{X}}, x) \simeq \begin{cases} \Omega_{1_{\text{X}}} X & x \text{ is in the unit component} \\ \emptyset & \text{otherwise} \end{cases}$$

Tensoring with $\mathcal{V} : \text{CAlg}(\text{Pr}^L)$ one obtains six functors landing in $\text{Mod}_{\mathcal{V}}(\text{Pr}^L)$ i.e. presentably \mathcal{V} -enriched categories.

Remark 1.20. We reiterate that all these six functor formalisms will be *strongly* symmetric monoidal. This will come into the story later in section 3.2.

¹⁰Standard notation would be some decorated version of \otimes , which we reserve in concrete settings for spectra.

Proof. We want to apply the universal property of the span 2-category (an easier version of [CLL25, Thm. A and B]) to $\text{An}_{\text{all}; \text{all}; \text{iso}}$ ¹¹.

To show this we need to establish the Beck–Chevalley condition for left Kan extension. This is evident from the object-wise formula. \square

Remark 1.21. There are more elaborate six functor formalisms of sheaves on (topological) spaces/locales. The locally constant sheaves only depend on the *shape* of spaces, which in nice cases is the same as (weak) homotopy type ([Lur17, Appx. A]). This doesn't hold for non-constant sheaves.

The six functor formalism of parametrised pointed spaces $\text{PShv}(-; \text{An}_*) \simeq \text{An}_*^{-12}$ allows us to deduce the following properties of Thom spaces.

Corollary 1.22 (Integration of external smash products) *Consider a pullback diagram in An .*

Let $p : W \rightarrow Z$ be the diagonal. For $\zeta : \text{vb}^X$ and $\xi : \text{vb}^Y$, we have

$$p_!(S_W^{\zeta \boxplus_Z \xi}) \simeq f_!(S_X^\zeta) \wedge_Z g_!(S_Y^\xi).$$

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ \downarrow g' & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

In particular, for any $X \xrightarrow{f} Y$ there holds: $f_!(S_X^{\zeta \boxplus_X f^ \xi}) \simeq f_!(S_X^\zeta) \wedge_Y S_Y^\xi$.*

Proof. The last assertion is a specialisation to $g = \text{id}_Y$. The general statement is a straightforward computation using the projection formula:

$$\begin{aligned} p_!(S_W^{\zeta \boxplus_Z \xi}) &\stackrel{\text{Remark 1.11}}{=} p_!(S_W^{g'^*(\zeta) \oplus_W f'^*(\xi)}) \\ &\stackrel{\text{Lemma 1.12}}{=} f'_! g'_!(g'^*(S_X^\zeta) \wedge_W f'^*(S_Y^\xi)) \\ &\stackrel{\text{proj. formula}}{=} f'_!(S_X^\zeta \wedge_X g'_! f'^*(S_Y^\xi)) \\ &\stackrel{\text{Beck–Chevalley}}{=} f'_!(S_X^\zeta \wedge_X f^* g_!(S_Y^\xi)) \\ &\stackrel{\text{proj. formula}}{=} f_!(S_X^\zeta) \wedge_Z g_!(S_Y^\xi). \end{aligned} \quad \square$$

These imply a plethora of properties for Thom spaces.

Exercise 3 Integrating the associated sphere bundle S_X^ζ along $X \xrightarrow{f} Y$ gives a notion of “relative” Thom construction. What does the above corollary say about them?

Exercise 4 (Stabilisation of vector bundles) Let $\varepsilon_X := X^* \varepsilon$ denote the rank 1 trivial bundle over X . By the stabilisation of $\zeta : \text{vb}^X$ we mean taking (internal) direct sums with ε_X . What does the corollary say about this? Explain why this justifies the following notation:

$$\text{Th}_X(\xi) =: \Sigma^\xi X_+.$$

2. Thom spectra, orientation and duality

In this section we'll introduce Thom spectra, a stabilised version of the Thom space we've seen in the last section, and see a few applications of Thom spectra in topology.

¹¹We chose $I = \text{all}$ so that the machine spits out *left* Kan extension as the $(-)_!$ functoriality.

¹²As we work with anima, we omit op from the notation.

Slogan Thom spectra are suspension (spectra) twisted by a vector bundle.

2.1. Thom spectra

Taking $\mathcal{V} = \text{Sp}$ in theorem 1.19 and tensoring up along $\Sigma^\infty : \text{An}_* \rightarrow \text{Sp}$, we obtain:

1. The stable six functors of parametrised spectra: $\text{Sp}^- : \text{An}^{\text{op}} \rightarrow \dots^{13} \text{Pr}_{\text{st}}^{\text{L}}$, with a stable version of Theorem 1.19 and Corollary 1.22. It has the fiberwise monoidal structure \otimes_X for every space and Day convolution monoidal structure \otimes^X for \mathbb{E}_n -spaces.
2. Stable Thom construction a.k.a. Thom spectra $\mathbb{S}_{\bar{X}}^X : \text{vb}_{\text{inj}}^X \rightarrow \text{Sp}^X$.

Exercise 5 State and argue for the analogues of the unstable results.

In the stable case, Thom spectra extends to virtual vector bundles.

Definition-Proposition 2.1 *There is a unique dashed arrow which fills the square below (compatibly for all X):*

$$\begin{array}{ccccc}
 \text{vb}^X & \hookrightarrow & \text{vb}_{\text{inj}}^X & \xrightarrow{\mathbb{S}_{\bar{X}}} & \text{An}_*^X \\
 \downarrow & & & & \downarrow \Sigma^\infty \\
 \text{kgl}^X & & & \xrightarrow{\mathbb{S}_{\bar{X}}} & \text{Sp}^X
 \end{array}$$

which we call the **stable sphere bundle** functor.

The **Thom spectrum** functor is the composition

$$\text{kgl}^X \xrightarrow{\mathbb{S}_{\bar{X}}} \text{Sp}^X \xrightarrow{X_i} \text{Sp}.$$

It sends $\xi : \text{kgl}^X$ to its Thom spectrum, which we denote by

$$\Sigma_+^{\infty+\xi} X := X_i(\mathbb{S}_{\bar{X}}^\xi) : \text{Sp}.$$

Proof. The morphism $\text{vb} \rightarrow \text{An}_* \rightarrow \text{Sp}$ is symmetric monoidal and lands in the $\otimes_{\mathbb{S}}$ -invertible objects, i.e. in the Picard group $\text{Pic}(\mathbb{S})$. The factorisation exists uniquely by the universal property of groupification. \square

Remark 2.2. Often one considers the rank 0 virtual bundle associated to a vector bundle ξ . More concretely this is classified by the morphism

$$\xi_0 : X \xrightarrow{\xi} \text{kgl} \simeq \mathbb{Z} \times \mathbf{BGL} \twoheadrightarrow 0 \times \mathbf{BGL} \hookrightarrow \text{kgl}.$$

Example 2.3 (Cobordism spectra) The Thom spectrum of the tautological rank 0 virtual vector bundle on \mathbf{BO} is denoted \mathbf{MO} . Initially introduced by R. Thom studying bordism theory, their homotopy groups are shown to be the rings of bordism classes of closed manifolds (resp. (stably) almost compact manifolds). These are susceptible to computation while also enjoying good abstract properties.

Moreover, for any tangential structure $\mathbf{BG} \rightarrow \mathbf{BO}$, the homotopy groups of the corresponding Thom spectrum \mathbf{MG} give the cobordism classes of manifolds with a stably normal G -structure.

On the other hand, the (ring) spectrum \mathbf{MU} carries a surprisingly rich structure, which is hugely important to the study of finer structures of spectra and the category of spectra. The related studies is conducted under the name of **Chromatic** Homotopy Theory.

The existence of these ring structure will be discussed later (Example 3.4).

¹³I'm too lazy.

2.2. Orientations of (virtual) vector bundles

Classically, an orientation of an n -dimensional \mathbb{R} -vector space V can be specified in many ways:

1. From the perspective of linear algebra/differential geometry, an orientation is a choice of connected component of the top-dimensional exterior power $\Lambda^n V$.
2. Topologically an orientation is a choice of generator of the local (co)homology at the origin: $H_n(V, V \setminus 0)$ or $H^n(V, V \setminus 0)$.

An orientation of a vector bundle is then a suitably parametrised version of such datum. This is exactly what Thom spectra are built to do!

In this section, we equip all Sp^X with *the fiberwise monoidal structure*.

Definition-Proposition 2.4 (A -orientation) *Let $A : \mathrm{Alg}(\mathrm{Sp}^X)$ be a parametrised ring over X and $\xi : \mathrm{kgl}^X$ a virtual bundle. For a morphism $\alpha : \mathbb{S}_X^\xi \rightarrow A$, the following are equivalent:*

1. *The induced (left) A -module morphism*

$$A \otimes_X \mathbb{S}_X^\xi \rightarrow A$$

is an isomorphism.

2. *For each point $1 \xrightarrow{x} X$ of X , the morphism*

$$\mathbb{S}^{\xi_x} \simeq x^*(\mathbb{S}_X^\xi) \xrightarrow{x^*(\alpha)} x^*(A) =: A_x$$

gives a degree $\mathrm{rk}_x \xi$ invertible element in the graded ring $\pi_\bullet(A_x)$.

If the above is satisfied, we call α an A -orientation of ξ .

For a ring $B : \mathrm{Alg}(\mathrm{Sp})$, orientations in the constant parametrised ring $\underline{B}_X := X^(B)$ are also called B -orientations.*

Remark 2.5. Orientations of vector bundles in a (non-zero) connective ring spectrum only exist when the vector bundle has constant rank o . For this reason one often defines orientations as orientations of the associated rank o virtual bundle.

Proof of the equivalence. To check that the morphism $A \otimes_X \mathbb{S}_X^\xi \rightarrow A$ in a functor category is an isomorphism, it suffices that it is objectwise an isomorphism, i.e. for every $x : X$

$$x^*(A \otimes_X \mathbb{S}_X^\xi) \simeq A_x \otimes_{\mathbb{S}} \mathbb{S}^{\xi_x} \simeq \Sigma^{\mathrm{rk}_x \xi} A_x \xrightarrow{\alpha_x} A_x$$

is an isomorphism, which is evidently equivalent to the second condition. □

In other words, orientations in a parametrised ring A witness a “relative constancy” of the free A -module twisted by a vector bundle.

Exercise 6 An orientations in a constant parametrised ring can also be given as the adjoint morphism $\tilde{\alpha} : \Sigma_+^{\infty+\xi} X = X_!(\mathbb{S}_X^\xi) \rightarrow B$. The map $\mathbb{S}^{\xi_x} \rightarrow B$ in the second condition coincides with the map

$$\mathbb{S}^{\xi_x} \simeq x^*(\mathbb{S}_X^\xi) \simeq X_! x_! x^*(\mathbb{S}_X^\xi) \xrightarrow{X_! \tilde{\alpha}} X_!(\mathbb{S}_X^\xi) \xrightarrow{\tilde{\alpha}} B,$$

where ε is the counit of the adjunction $x_! \dashv x^*$.

We're one step away from the Thom isomorphism now. Consider an orientation in a *constant* parametrised ring \underline{A}_X and apply integration to the isomorphism $\underline{A}_X \otimes_X \mathbb{S}_X^\xi \xrightarrow{\sim} \underline{A}_X$. The projection formula now identifies the resulting isomorphism.

Proposition 2.6 (Thom isomorphism) *Consider a vector bundle $\xi : \text{kgl}^X$, ring $A : \text{Alg}(\text{Sp}^Y)$ and a map $X \xrightarrow{f} Y$. Any $f^*(A)$ -orientation of ξ , induces an isomorphism*

$$A \otimes_Y f_!(\mathbb{S}_X^\xi) \xrightarrow{\sim} A \otimes_Y f_!(\mathbb{S}_X^0).$$

In the case $Y = 1$ we recover the classical A -Thom isomorphism

$$A \otimes_{\mathbb{S}} \Sigma_+^{\infty+\xi} X \xrightarrow{\sim} A \otimes X. \quad \square$$

Our proof is entirely parallel to the common textbook approach using the theorem of Leray–Hirsch. In fact we obtain a version of Leray–Hirsch almost for free by reinterpret the definition 2.4. See Appendix A for the statement.

Example 2.7 (Orientations classically) Classically, an orientation of a rank n vector bundle $\xi : \text{vb}^M$ is reduction of its structure group from $\mathbf{BO}(n)$ to $\mathbf{BSO}(n)$, i.e. a dashed lift in the following diagram

$$\begin{array}{ccccccc}
 & & \mathbf{BSO}(n) & \rightarrow & \mathbf{BSO} & \xrightarrow{\quad} & \mathbf{B1} \\
 & \nearrow & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 M & \xrightarrow{\xi} & \mathbf{BO}(n) & \rightarrow & \mathbf{BO} & \rightarrow & \mathbf{BGL}_1(\mathbb{S}) \rightarrow \mathbf{BGL}_1(\mathbb{Z}) \simeq \mathbf{B}\mathbb{Z}/2
 \end{array}$$

As the squares in the diagrams are pullbacks, orientations *in the classical sense* are the same as null homotopies of the composition $M \rightarrow \mathbf{B}\mathbb{Z}/2$, which one can show is the first Stiefel–Whitney class of ξ .

On the other hand, a null homotopy of $M \rightarrow \mathbf{BGL}_1(\mathbb{Z})$ is exactly a \mathbb{Z} -orientation *in our sense* of the rank 0 virtual bundle ξ_0 , i.e. a witness to the constancy of the associated parametrised \mathbb{Z} -module.

Exercise 7 At a Topology II lecture, the professor defines orientation of manifolds or vector bundles over an arbitrary ring, but the ring has never been something other than $\mathbb{Z}/2$ or \mathbb{Z} .

Somewhere during the semester, the annoying student brought up a question: “Why don’t we work with another ring, like $\mathbb{Z}/3$?”

“Oh, then it’s actually the same as orienting over \mathbb{Z} .” The professor then moves on with the lecture, and so does everybody in the room.

Two years later, having absorbed into herself the language of homotopy theory, the student realises that the professor was right all along!

1. Every vector bundle has a unique $\mathbb{Z}/2$ -orientation.
2. The datum of a $\mathbb{Z}/3$ - or $\mathbb{Z}/4$ -orientation is the same as a \mathbb{Z} -orientation.
3. Does the same hold for $\mathbb{Z}/5$ -orientations?

2.3. Duality, Atiyah and Poincaré

We borrow the following result from differential topology.

Theorem 2.8 (Atiyah duality [Rez13]) For a closed manifold M and a virtual vector bundle $\xi : \text{kgf}^M$, the Spanier-Whitehead dual of its Thom spectrum is equivalent to

$$(\Sigma_+^{\infty+\xi} M)^\vee \simeq \Sigma_+^{\infty-\xi-TM} M,$$

where TM is the tangent bundle of M .

More generally, for a compact manifold $(M, \partial M)$, there holds

$$(\Sigma^\infty M/\partial M)^\vee \simeq \Sigma_+^{\infty-TM} M. \quad \square$$

Corollary 2.9 (Poincaré duality) Let $(M, \partial M)$ be a compact m -manifold. Any orientation of M induces isomorphisms

$$H^\bullet(M, \partial M; \mathbb{Z}) \simeq H_{m-\bullet}(M; \mathbb{Z}) \text{ and } H_\bullet(M, \partial M; \mathbb{Z}) \simeq H^{m-\bullet}(M; \mathbb{Z}).$$

Proof. An orientation of a manifold M is a \mathbb{Z} -orientation of $TM - m$. This induces (and is induced by) a \mathbb{Z} -orientation of $m - TM$.

Atiyah duality 2.8 and Thom isomorphism 2.6 implies

$$(\mathbb{Z} \otimes_{\mathbb{S}} (M/\partial M))^\vee \simeq \mathbb{Z} \otimes_{\mathbb{S}} \Sigma_+^{\infty-m+(m-TM)} M \simeq \mathbb{Z} \otimes_{\mathbb{S}} \Sigma_+^{\infty-m} M.$$

The result follows from taking homology. □

3. Rigidity of parametrised spectra and monoidality of Thom spectra

This section is motivated by my attempt using our language to illustrate a part of the following result of Blumberg–Cohen–Schlichtkrull:

Theorem 3.1 (Blumberg–Cohen–Schlichtkrull [BCS10, Thm. 1], [KN, Lem. 4.4]) Let $X \xrightarrow{\xi} \text{Pic}(\mathbb{S})$ be a map of grouplike \mathbb{E}_n -spaces. Then the Thom spectrum $\Sigma_+^{\infty+\xi} X$ is an \mathbb{E}_n -ring and there is an equivalence of \mathbb{E}_{n-1} -rings ($n \geq 1$)

$$\text{THH}(\Sigma_+^{\infty+\xi} X) \simeq \Sigma_+^{\infty+L^\eta(\mathbf{B}\xi)} \mathbf{L}B X,$$

where $L^\eta(\mathbf{B}\xi)$ is the map from the free loop space

$$\mathbf{L}B X \xrightarrow{\mathbf{L}B\xi} \mathbf{L}B\text{Pic}(\mathbb{S}) \simeq \text{Pic}(\mathbb{S}) \times \mathbf{B}\text{Pic}(\mathbb{S}) \xrightarrow{1 \otimes \eta} \text{Pic}(\mathbb{S}).$$

This result is really a combination of two separate results: There is the symmetric monoidality of a suitable Thom spectrum functor, and there is the whole machine of THH, that it lowers commutativity by one level, that it's preserved under monoidal functors and, most importantly, the computation of $\text{THH}(G)$ in An as $\mathbf{L}B G$.

The part regarding THH is far beyond the scope of these notes. (See [KN, Sec. 8] for the computation.) We'll only treat the monoidality part, which classically is due to Lewis [Lew86, Thm. 7.1].

3.1. Symmetric monoidality of Thom construction

The following will be taken on faith.

Proposition 3.2 ([Ram26, Cor. D]) *The straightening equivalence $\text{An}/_{\text{Pic}(\mathbb{S})} \rightarrow \text{An}^{\text{Pic}(\mathbb{S})}$ can be made symmetric monoidal, where the left hand side is endowed with the slicing monoidal structure [Lur17, Sec. 2.2.2] and the right hand side with Day convolution.* \square

Definition-Proposition 3.3 (The classical Thom spectrum functor) *The inclusion $\text{Pic}(\mathbb{S}) \hookrightarrow \text{Sp}$ is symmetric monoidal. By the Yoneda lemma, this induced a (cocontinuous) symmetric monoidal functor*

$$\Theta : \begin{array}{ccc} \text{An}/_{\text{Pic}(\mathbb{S})} & \longrightarrow & \text{Sp} \\ (X \xrightarrow{\xi} \text{Pic}(\mathbb{S})) & \longmapsto & \Theta_X \xi \end{array}$$

which we call the **classical Thom spectrum** functor. Its unique factorisation through the stabilisation will be denoted $\tilde{\Theta} : \text{Sp}^{\text{Pic}(\mathbb{S})} \rightarrow \text{Sp}$.

For each $X \xrightarrow{\xi} \text{Pic}(\mathbb{S})$, we have an equivalence

$$\Sigma_+^{\infty+\xi} X \simeq \Theta_X \xi.$$

Hence we know that this construction coincides with the Thom spectrum functor we've come to know and love.

Example 3.4 (Cobordism spectra again) Consider the Whitehead tower of \mathbf{BO} .

$$\dots \rightarrow \mathbf{BString} \rightarrow \mathbf{BSpin} \rightarrow \mathbf{BSO} \rightarrow \mathbf{BO}$$

As $\mathbf{BO} \rightarrow \text{Pic}(\mathbb{S})$ is an \mathbb{E}_∞ -map, all the above are \mathbb{E}_∞ -space over $\text{Pic}(\mathbb{S})$. Consequently, their Thom spectra $\mathbf{MString}$ etc. are all \mathbb{E}_∞ -rings.

Accordingly I should've called our functor $\mathbf{M}\Omega$, so that $\Theta(\mathbf{BG} \rightarrow \text{Pic}(\mathbb{S})) = \mathbf{MG}$ agrees with the classical notation.

Sketchy Handwaving for 3.3. By unstraightening $X \xrightarrow{f} \text{Pic}(\mathbb{S}) : \text{An}/_{\text{Pic}(\mathbb{S})}$ corresponds to $f_!(1_X) : \text{An}^{\text{Pic}(\mathbb{S})}$. We may work with its stabilisation $f_!(\mathbb{S}_X^0)$ instead. Note that $f_!(1_X)$ is an \mathbb{E}_n -algebra precisely if X is an \mathbb{E}_n -space over $\text{Pic}(\mathbb{S})$.

The goal is then to find an equivalence:

$$\tilde{\Theta}(f_!(\mathbb{S}_X^0)) \simeq X_!(\mathbb{S}_X^f). \quad (1)$$

The inclusion $\text{Pic}(\mathbb{S}) \hookrightarrow \text{Sp}$ gives the *tatutological sphere bundle* $\gamma : \text{Sp}^{\text{Pic}(\mathbb{S})}$. Since lax symmetric monoidal functors are exactly the monoids under Day convolution, this makes γ a commutative algebra under Day convolution.

The sphere bundle of $X \xrightarrow{f} \text{Pic}(\mathbb{S})$ is exactly $f^*(\gamma)$, hence the Thom spectrum of f in our sense is

$$\begin{aligned} X_!(\mathbb{S}_X^f) &\simeq (\text{Pic}(\mathbb{S}))_!(f_! f^*(\gamma)) \\ &\simeq (\text{Pic}(\mathbb{S}))_!(f_!(\mathbb{S}_X^0) \otimes_{\text{Pic}(\mathbb{S})} \gamma), \end{aligned} \quad (2)$$

where the second equality follows from the projection formula. (Beware of the *fiberwise* tensor product!)

We claim the right hand side of (2) is exactly $\tilde{\Theta}(f_!(\mathbb{S}_X^0))$.

Claim: For any $X : \mathbf{An}$ the category \mathbf{Sp}^X is self dual. For any $m : \mathbf{Sp}^X$ viewed as an element of the dual corresponding to $M : \mathbf{Sp}^X \rightarrow \mathbf{Sp}$, there holds

$$M \simeq X_!(- \otimes_X m).$$

Moreover, the functor $\tilde{\Theta}$ corresponds to γ in this sense. Hence the functor $\Theta : \mathbf{An}/\mathbf{Pic}(\mathbb{S}) \rightarrow \mathbf{Sp}$ is indeed given by

$$\Theta(X \xrightarrow{f} \mathbf{Pic}(\mathbb{S})) \simeq \tilde{\Theta}(f_!(\mathbb{S}_X^0)) \simeq (\mathbf{Pic}(\mathbb{S}))_!(f_!(\mathbb{S}_X^0) \otimes_{\mathbf{Pic}(\mathbb{S})} \gamma). \quad \square$$

The claim about duality and evaluation having the stated form is inspected in the next section.

3.2. Rigid six functors (Recruiting a better name)

Recall that a six functor formalism on is a (product-preserving?) *lax* symmetric monoidal functor defined on a span 2-category.

Definition 3.5 (Recruiting a better name) A stable six functor formalism $D : \mathbf{C}^{op} \rightarrow^{E;I,P} \mathbf{Pr}_{st}^L$ is called **rigid** if it is strongly symmetric monoidal.

Example 3.6 The stable six functor formalism of parametrised spectra $D : \mathbf{An}^{op} \rightarrow \mathbf{Pr}_{st}^L$ is symmetric monoidal.

We illustrate below how this gives self duality of \mathbf{Sp}^X and give a description of the evaluation functor, this will conclude the argument of the previous section.

Lemma 3.7 (Frobenius structure in the span category) The Cartesian monoidal structure on \mathbf{An} induces on each $X : 2\mathbf{Span}(\mathbf{An})$ the structure of a Frobenius algebra.

The structure spans for the (co)unit, (co)multiplication and (co)evaluation are:

$$\begin{array}{lll} \eta_X = \begin{array}{c} X \\ \swarrow \scriptstyle{x} \quad \searrow \scriptstyle{=} \\ 1 \quad X \end{array} & \mu_X = \begin{array}{c} X \\ \swarrow \scriptstyle{\Delta} \quad \searrow \scriptstyle{=} \\ X \times X \quad X \end{array} & \text{ev}_X = \varepsilon_X \circ \mu_X = \begin{array}{c} X \\ \swarrow \scriptstyle{\Delta} \quad \searrow \scriptstyle{x} \\ X \times X \quad 1 \end{array} \\ \varepsilon_X := \begin{array}{c} X \\ \swarrow \scriptstyle{=} \quad \searrow \scriptstyle{x} \\ X \quad 1 \end{array} & \delta_X = \begin{array}{c} X \\ \swarrow \scriptstyle{=} \quad \searrow \scriptstyle{\Delta} \\ X \quad X \times X \end{array} & \text{coev}_X = \delta_X \circ \eta_X = \begin{array}{c} X \\ \swarrow \scriptstyle{x} \quad \searrow \scriptstyle{\Delta} \\ 1 \quad X \times X \end{array} \end{array}$$

A rigid six functor formalism $D : 2\mathbf{Span}(\mathbf{An}) \rightarrow \mathbf{Pr}_{st}^L$, establishes each $D(X)$ with fiber-wise monoidal structure as a Frobenius algebra in \mathbf{Pr}_{st}^L .

Remark 3.8. In the span 2-category $2\mathbf{Span}(\mathbf{An}_{\text{all},\text{all,iso}})$, there are adjunctions $\varepsilon \dashv \eta$ and $\delta \dashv \mu$. This is a dual notion of *rigid algebra* in a 2-category [KNP, Sec. 4.2, 4.3].

As the six functor formalism of parametrised spectra is rigid, we obtain:

Corollary 3.9 The evaluation $\mathbf{Sp}^X \otimes \mathbf{Sp}^X \simeq (\mathbf{Sp}^X)^\vee \otimes \mathbf{Sp}^X \rightarrow \mathbf{Sp}$ is given by

$$\text{ev}_X(M, N) \simeq X_!(M \otimes_X N). \quad \square$$

Proof of Lemma 3.7. The algebra and coalgebra structures is known to us. The remaining identities in the definition of a Frobenius algebra is equivalent to the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times X \\
\Delta \downarrow \ulcorner & & \downarrow \Delta \times 1 \\
X \times X & \xrightarrow{1 \times \Delta} & X \times X \times X
\end{array}$$

being a pullback. It's classical that a Frobenius structure provides a self-dual datum on the algebra, where the (co)evaluation are as claimed. \square

Exercise 8 Fill in the details if you haven't seen it before.

Exercise 9 (The gap in the proof of 3.3) Show that the duality pairing induced by the Frobenius structure coincides with Yoneda extension. In other words, the following diagram commutes (where the duality is in Pr_{st}^L , so $(-)^{\vee} = \text{Fun}^L(-, \text{Sp})$).

$$\begin{array}{ccc}
\text{Sp}^X \otimes \text{Sp}^X & \xrightarrow{(\text{Yoneda ext.}) \otimes \text{iso}} & (\text{Sp}^{X^{\text{op}}})^{\vee} \otimes \text{Sp}^{X^{\text{op}}} \\
\searrow & & \swarrow \text{ev} \\
& \text{Sp} &
\end{array}$$

Remark 3.10 (Monoidality of duality (prospective)). A feasible extension of the above claim would be to further identify the monoid structure on γ with the lax monoidality on the functor it represents, namely that $F_m := X_!(- \otimes_X m)$ shall inherit a structure of a lax monoidal functor from a convolution monoid structure on m .

Heuristically, the lax structure maps we want are like $F_m(x) \otimes_{\mathbb{S}} F_n(y) \rightarrow F_{m \otimes_X n}(x \otimes^X y)$, i.e.

$$X_!(x \otimes_X m) \otimes_{\mathbb{S}} X_!(y \otimes_X n) \rightarrow X_!((x \otimes^X y) \otimes_X (m \otimes_X n)),$$

Monoidality of $X_!$ with respect to convolution tells us this is equivalent to

$$X_!((x \otimes_X m) \otimes^X (y \otimes_X n)) \rightarrow X_!((x \otimes^X y) \otimes_X (m \otimes_X n)).$$

I expect such a structure to come from the “cocommutative- \mathbb{E}_n -bialgebra” structure on X , namely a compatibility of the \mathbb{E}_n -monoid structure with the canonical Cartesian \mathbb{E}_{∞} -comonoid structure. For instance, the above pairing should come from the following 2-morphism in the span 2-category:

$$\begin{array}{ccccc}
& & X^{\times 2} & & \\
& \delta_{X^{\times 2}} \swarrow & \downarrow m & \searrow m & \\
X^{\times 4} & & X & & X \longrightarrow 1 \\
& \swarrow m \times m & \downarrow \delta_X & \searrow \delta_X & \\
& & X^{\times 2} & &
\end{array}$$

A precise formulation would be desirable, an attempt of mine is recorded in Appendix B.

3.3. The canonical orientation of Thom ring spectra

I type this section up under the impression that it's folklore. The result is at least a decade old and can be found in a more refined form in e.g. [AB19, Lem. 3.15, Cor. 3.17].

For a coherent storytelling I'm assuming that the equivalence $\Theta_X(\gamma) \simeq X_!(\mathbb{S}_X^{\gamma})$ respects the algebra structures on both sides. The reader unconvinced by the sketchy remark above might want

to replace all instances of $\tilde{\Theta}_X(\gamma)$ in this section with the latter. I'd like to keep their distinction in notation as a reminder of the role of $\tilde{\Theta}_X(\gamma)$ as the ring for the orientation resp. $X_!(\mathbb{S}_X^\gamma) \simeq \Sigma_+^{\infty+\gamma} X$ as the “orientation classifier”.

We now cook up a mixture of the material we've discussed of the previous section. To prepare the reader for potential confusion, I remind the reader that an \mathbb{E}_n -space X equips Sp^X with two monoidal structures:

1. The *fiberwise* tensor product, denoted \otimes_X , is symmetric monoidal. Pullback is symmetric monoidal, integration is oplax and the projection formula holds along *all* maps.
2. The *convolution*, denoted \otimes^X , is \mathbb{E}_n -monoidal. Integration is \mathbb{E}_n -monoidal and pullback is lax along \mathbb{E}_n -maps.
3. The two coincide on $X = 1$ and is just the tensor product of spectra. In particular, the fiberwise monoidal unit is the pullback of \mathbb{S} along $X \rightarrow 1$, denoted \mathbb{S}_X^0 .

On the other hand, the convolution unit δ^X is the integration of \mathbb{S} along the unit morphism $1 \xrightarrow{\eta} X$. In particular it is zero except on the unit component, where it takes value $\mathbb{S}[\Omega X]$.

Consider a connected¹⁴ \mathbb{E}_n -space X with a \mathbb{E}_n -map $X \xrightarrow{\gamma} \mathrm{Pic}(\mathbb{S}) \subseteq \mathrm{Sp}$. As we have seen, this is a \mathbb{E}_n -monoid under convolution \otimes^X .

Question 1 (Does a monoidal Thom spectrum “orient itself?”) Is the isomorphism 3.3 $\Sigma_+^{\infty+\gamma} X \xrightarrow{\sim} \tilde{\Theta}_X(\gamma)$ a $\tilde{\Theta}_X(\gamma)$ -orientation of γ ?

Our connectedness assumption implies that this is an \mathbb{S}^0 -bundle γ . Moreover, the condition that the map $\mathbb{S}_X^\gamma \xrightarrow{\varepsilon} X^*(\tilde{\Theta}_X(\gamma))$ be a unit of the ring can be checked fiberwise. The question reduces to whether

$$\mathbb{S}^0 \simeq \eta^* \mathbb{S}_X^\gamma \rightarrow \tilde{\Theta}_X(\gamma) \tag{3}$$

is a unit in the right hand side. But ε is a map of rings under *convolution*¹⁵:

Lemma 3.11 *In an adjunction $C \overset{L}{\underset{R}{\rightleftarrows}} D$ between \mathcal{O} -monoidal categories, where L is \mathcal{O} -monoidal, the right adjoint R is canonically lax \mathcal{O} -monoidal and the counit and unit are \mathcal{O} -monoidal transformations.*

Proof. This is essentially [Lur17, Cor. 7.3.2.7]. Beware that Cor. 7.3.2.7 is not correct as stated, as the left adjoint F needs to be \mathcal{O} -monoidal in order for Prop. 7.3.2.6 to be applicable, as in the proof presented in loc. cit. □

Exercise 10 Extract the coherence datum for the monoidality of the (co)unit transformation from the relative adjunction of [Lur17, Sec. 7.3.2].

¹⁴Group-like might be sufficient. Most notable non-connected case is classifier of stable vector bundles $ku \simeq \mathbb{Z} \times \mathbf{BU}$, which produces the periodic complex bordism spectrum MUP. The reader can take this as an exercise, or I'll attempt it in a later version.

¹⁵I warn the reader that this is a map of rings in the *convolution* monoidal structure, which is not the fiberwise one in which we use to work with orientations.

Hence the lax monoidality of η^* produces maps

$$\begin{array}{ccc} \eta^* \delta^X \simeq \mathbb{S}[\Omega X] & \longrightarrow & \tilde{\Theta}_X(\gamma) \\ \uparrow & \searrow & \uparrow \eta^{*\varepsilon} \\ \mathbb{S} & \xrightarrow{\sim} & \eta^* \mathbb{S}_X^\gamma \end{array}$$

We conclude that ε is even fiberwise *the* unit, and hence an orientation.

Theorem 3.12 (Thom ring spectra self-orient) *For a map $X \xrightarrow{\xi} \text{Pic}(\mathbb{S})$ of \mathbb{E}_1 -spaces, there is a canonical $\tilde{\Theta}_X \xi$ -orientation of ξ , given by the identity map.* \square

Corollary 3.13 (Of Thom isomorphism 2.6) *In the setting of 3.12, one has an isomorphism of spectra*

$$\tilde{\Theta}_X \xi \otimes_{\mathbb{S}} \tilde{\Theta}_X \xi \simeq \tilde{\Theta}_X \xi \otimes X.$$

Question 2 Is the Thom isomorphism an isomorphism of algebras? In well known cases this is true. This is again a compatibility question of convolution and fiberwise tensoring.

Example 3.14 One has $\text{MU} \otimes_{\mathbb{S}} \text{MU} \simeq \text{MU} \otimes \text{BU}$. In particular $\mathbb{Z} \otimes_{\mathbb{S}} \text{MU} \simeq \mathbb{Z} \otimes \text{BU}$.

In fact one has $(\text{MU} \otimes_{\mathbb{S}} \text{MU})_{\bullet} \simeq \text{MU}_{\bullet}[\mathfrak{b}_n]_{n \geq 1}$, where $|\mathfrak{b}_n| = 2n$ lift generators of $(\mathbb{Z} \otimes \text{BU})_{\bullet} \simeq \mathbb{Z}[\mathfrak{b}_n]$.

3.3.1. Story unfinished: MU and complex orientations

I've decided to not expand out this section, as it deserves yet another proper chapter, which would make the document grow way beyond its intended purpose. Some intended topics are: [Complex orientable rings and the Thom isomorphism \$\mathbb{E} \otimes_{\mathbb{S}} \text{MU} \simeq \mathbb{E} \otimes \text{BU}\$, even rings are orientable..](#)

A. The Leray–Hirsch theorem

The Thom isomorphism is an instance of a constancy criterion for parametrised modules. We can package this criterion into a theorem, from which we can recover the classical theorem of Leray–Hirsch.

We'll start by fixing some notations and recalling how homology and cohomology look like in our setting.

For a (ring) spectrum $A : \text{Sp}$ and a pair of spaces $(X, X_0) : \text{An}^{[1]}$, we denote the spectrum corresponding to their relative A -homology by

$$A[X, X_0] := A \otimes_{\text{An}_*} (X/X_0, *) \simeq A \otimes_{\mathbb{S}} \Sigma^{\infty}(X/X_0).$$

In other words $\pi_{\bullet} A[X, X_0] =: A_{\bullet}(X, X_0)$. We write $A[X]$ for when $X_0 = \emptyset$, giving the unreduced homology.

If $(E, e) : \text{An}_*^{\mathbb{B}}$ is a parametrised pointed space, then we can define the parametrised reduced A -homology spectrum by

$$\underline{A}_{\mathbb{B}}[E, e] := \underline{A}_{\mathbb{B}} \otimes_{\text{An}_*^{\mathbb{B}}} (E, e) : \text{Sp}^{\mathbb{B}}.$$

By the projection formula we know that $B_!(\underline{A}_B[E, e]) \simeq A[E, e]$ gives as homotopy groups the A -homology of the total space E relative the section e . An analogous statement holds for A -cohomology, which we summerise as follows:

$$\begin{aligned} A_\bullet(E, e) &\simeq \pi_\bullet(A[E, e]) \\ &\simeq \pi_\bullet B_!(\underline{A}_B[E, e]) \\ A^\bullet(E, e) &\simeq \pi_\bullet \text{map}_{A\text{-mod}}(A[E, e], A) \\ &\simeq \pi_\bullet \text{map}_{\underline{A}_B\text{-mod}}(\underline{A}_B[E, e], \underline{A}_B). \end{aligned}$$

We now state our algebraic version of Leray–Hirsch theorem, whose proof we omit, as it is an easy modification of the argument around Proposition 2.6 and Definition-Proposition 2.4.

Theorem A.1 (Algebraic) Leray–Hirsch in homology) *Let A be a ring and $M_0 : A\text{-Mod}$. Consider a parametrised A -module $M : A\text{-Mod}^B$ over $B : \text{An}$, together with a map $B_!M \xrightarrow{\alpha} M_0$ of A -modules.*

*If for each point $b : B$, the natural map $M_b \rightarrow B_!M \xrightarrow{\alpha} M_0$ is an isomorphism, then the adjoint map of parametrised A -modules $M \rightarrow B^*M_0$ is an isomorphism. This isomorphism integrates to an isomorphism $B_!M \simeq A[B] \otimes_A M_0$. \square*

Remark A.2. The cohomological situation is similar and also interesting, but note that the right adjoint B_* is lax monoidal and hence lifts to $(A[B])^\vee$ -modules.

This difference is only apparent, as the homology clearly are $A[B]$ -comodules. It is partly due to our mathematical culture that we perceive them less, and partly because coalgebraic structures don't survive to the homotopy groups.

Exercise 11 (Algebraic Leray-Hirsch in cohomology) Study the analogous statement on $M_0 \rightarrow B_*(M)$.

We now deduce the classical Leray–Hirsch theorem, after introducing some notations.

Corollary A.3 (Leray–Hirsch) *Let A be a commutative ring spectrum and $(E, e) : \text{An}_*^B$ a pointed bundle over a connected base B . Suppose moreover that E has fiberwise compact A -homology, meaning that (each) $A[E_b, e_b]$ is a perfect A -module. Choose a base point of B to fix a distinguished fiber $(F, f_0) := (E_0, e_0)$.*

Given relative A -cohomology classes $c_1, \dots, c_r : A^\bullet(E, e)$ whose restrictions to the fiber $A^\bullet(F, f_0)$ freely generate $A^\bullet(F, f_0)$ as an A -module, there is an induced isomorphism

$$A_\bullet(E, e) \xrightarrow{\sim} A_\bullet(B) \otimes_A A_\bullet(F, f_0).$$

Similarly in A -cohomology, there is an isomorphism of $A^\bullet(B)$ -modules.

$$A^\bullet(E, e) \xrightarrow{\sim} A^\bullet(B) \otimes_A A^\bullet(F, f_0).$$

In particular if the fiber F is a finite, or more generally a compact anima, then it has compact A -homology.

Proof. We consider the homological case, the cohomological case is easier and left as an exercise. The reader may find the projection formula on mapping objects (Proposition 1.15, part 4) useful (in the present case $f^! \simeq f^*$).

Let $M : A\text{-Mod}^B$ be the fiberwise A -homology of E , i.e. $M_b := A[E_b, e_b]$.

The given cohomology classes induce a morphism of A -modules

$$B_!M \xrightarrow{c} \bigoplus_{i=1}^r \Sigma^{|\mathcal{C}_i|} A.$$

The compactness of $A[F, f_0]$ implies that the map

$$A[F, f_0] \rightarrow B_!M \xrightarrow{c} \bigoplus_{i=1}^r \Sigma^{|\mathcal{C}_i|} A$$

is dual to the cohomology isomorphism

$$A^\bullet(F, f_0) \simeq \bigoplus_{i=1}^r A^{\bullet-|\mathcal{C}_i|}.$$

and is therefore an isomorphism to begin with. Theorem A.1 now applies to give the conclusion for homology. \square

Exercise 12 Prove Leray–Hirsch for cohomology. Is the compact homology assumption necessary here?

B. (Cocommutative) bialgebra structures

This appendix is a short sketch of a biased view on (cocommutative) bialgebra structures. The original motivation is to convince myself that the self-duality given by the fiberwise monoidal structure is compatible with the convolution on $D(X)$ and $D(X)^\vee$. An ad-hoc approach would be to identify the self-duality as Yoneda extension, but I'd like to see if I can make it work within the framework of six functor formalisms.

In a symmetric monoidal category \mathcal{C} , one would expect an \mathcal{O} -co- \mathcal{C} -bialgebra to be given by an \mathcal{O} -algebra in the category of \mathcal{C} -coalgebras, where the latter is equipped with a symmetric monoidal structure lifting the one on \mathcal{C} . Unfortunately, our usual language for operads as operator categories $\mathcal{O}^\otimes \rightarrow \mathbb{E}_\infty$ is somewhat inconvenient for comonoidal structures. There is the dual (partially opposite) notion of anti-operads which is dually unwieldy for our current objective. Moreover, we'd expect \mathcal{C} -coalgebras in \mathcal{O} -algebras to be the same \mathcal{O} -algebras in \mathcal{C} -coalgebras, which would undoubtedly be an annoying nightmare.

In our lucky situation where the bialgebras are cocommutative, the tensor product of cocommutative coalgebras is Cartesian ([Lur17, Prop. 3.2.4.7]). In this situation we have the Lawvere theory available as an alternative encoding of algebraic structures.

We will not survey Lawvere theories here. In a nutshell, a Lawvere theory T is a category where every object is a finite coproduct of a fixed object¹⁶ $t_1 : T$, and (Cartesian) T -algebras in a category \mathcal{C} are product-preserving functors $T^{\text{op}} \rightarrow \mathcal{C}$. One should think of the objects of T as the “finite free T -algebras”, which encode all the operation on a “ T -algebra”.

Here are two classical examples, the “trivial” theory and the \mathbb{E}_∞ -theory.

Proposition B.1 *For a category \mathcal{C} with products, there are equivalences:*

- $\text{CCog}(\mathcal{C}) := \text{Fun}^\times(\text{Fin}^{\text{op}}, \mathcal{C}) \simeq \mathcal{C}$ (every object is canonically a Cartesian coalgebra).

¹⁶I might want the object to be non-final.

- $\text{CAlg}(\mathcal{C}) \simeq \text{Fun}^\times(\text{Span}(\text{Fin})^{\text{op}}, \mathcal{C})$.

Sketch of proof. The first equivalence follows from Fin being the category obtained from the contractible category $\mathbf{1}$ by freely adjoining coproducts. I'll let the reader convince themselves that this does encode the structure of a Cartesian coalgebra.

The second equivalence is more classical. See [Cra, Sec. 5.3]. □

Remark B.2. In $\text{Span}(\text{Fin})$, the forward morphisms are a copy of Fin and encode the diagonals (duplication) and projections, and witness the decomposition of finite sets as a coproduct of singletons. (They're biproducts in $\text{Span}(\text{Fin})$). The additive structure is encoded by the backwards morphisms.

Indeed, the Cartesian coalgebra structure is fused into any Lawvere theory, but our goal is to separate the two.

We now have two equivalent options to proceed. Either we employ the first equivalence to obtain $\text{Fun}^\times(\mathbb{T}^{\text{op}}, \mathcal{C}) \simeq \text{Fun}^\times(\mathbb{T}^{\text{op}}, \text{Fun}^\times(\text{Fin}^{\text{op}}, \mathcal{C}))$, or we apply the tensoring of Fin on any Lawvere theory \mathbb{T} . In either case, we obtain:

Corollary B.3 For a Cartesian monoidal category \mathcal{C} , we have

$$\text{CAlg}(\mathcal{C}) \simeq \text{Fun}^{\times, \times}(\text{Span}(\text{Fin})^{\text{op}} \times \text{Fin}^{\text{op}}, \mathcal{C}).$$
□

Applying a lax monoidal functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{S}$, where \mathcal{S} is a monoidal 2-category, we would obtain a functor $\text{Span}(\text{Fin}) \times \text{Fin} \rightarrow \mathcal{S}$.

Recall our goal of encoding the compatibility of fiberwise tensors and convolution on $D(X) = \text{Sp}^X$ for an \mathbb{E}_∞ -space X . It is now obvious what to do: we pass to the left adjoints in the $\text{Span}(\text{Fin})$ -direction, at the cost of turning commuting squares into oplax ones. Beware that D is strongly monoidal in this case.

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \ddots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \mathbf{1} & \xrightarrow{\eta!} & D(X)^{\otimes(1\cdot 2)} & \xleftarrow{\mu!} & D(X)^{\otimes(2\cdot 2)} & \xleftarrow{\quad} & \dots \\
 \delta^* \downarrow & \nearrow & \delta^* \downarrow & \nwarrow & \delta^* \downarrow & & \\
 \mathbf{1} & \xrightarrow{\eta!} & D(X)^{\otimes(1\cdot 1)} & \xleftarrow{\mu!} & D(X)^{\otimes(2\cdot 1)} & \xleftarrow{\quad} & \dots \\
 \varepsilon^* \uparrow & \nwarrow & \varepsilon^* \uparrow & \nearrow & \varepsilon^* \uparrow & & \\
 \mathbf{1} & \xrightarrow{\eta!} & \mathbf{1} & \xleftarrow{\mu!} & \mathbf{1} & \xleftarrow{\quad} & \dots
 \end{array}$$

Example B.4 The upper right 2-cell is a transformation $(x \otimes_X m) \otimes^X (y \otimes_X n) \rightarrow (x \otimes^X y) \otimes_X (m \otimes^X n)$, which we mentioned in section 3.2.

More generally there is a transformation

$$\bigotimes_{\mathbb{S}}^m \left(\bigotimes_X^n x_{ij} \right) \rightarrow \bigotimes_X^n \left(\bigotimes_{\mathbb{S}}^m x_{ij} \right),$$

whose meaning and possible applications aren't yet clear to me at the moment.

Exercise 13 (for me as well) Can I deduce from the $n = 2$ case above that the self-duality $\text{Sp}^X \rightarrow \text{Fun}^L(\text{Sp}^X, \text{Sp})$ is (at least lax) monoidal, with both sides equipped with (Day) convolution?

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